April 10: Quasi-unmixedness and Ratliff'sTheorem, part 3

Asymptotic sequences

We can now prove Theorem H3.

Part (i): follows immediately from Theorem M3.

For example, suppose $x_1, \ldots, x_r$ is an asymptotic sequence.

If they do not remain an asymptotic sequence in $\widehat{R}$, then for some $j \leq r$, there exists $Q \in \overline{A^*}((x_1, \ldots, x_{j-1})\widehat{R})$ with $x_j \in Q$.

By Theorem M3, $P = Q \cap R$ belongs to $\overline{A^*}((x_1, \ldots, x_{j-1})R)$.

Since $x_j \in P$, this is a contradiction. The converse is similar.

Asymptotic sequences

Part (ii): The proof is similar to part (i), only one uses Corollary O3.

Suppose $x_1, \ldots, x_r$ is an asymptotic sequence. Let $q \subseteq R$ be a minimal prime, and maintain the notation from Corollary O3.

If the x_i do not remain an asymptotic sequence in R_q, then for some $j \leq r$, there exists $Q_q \in \overline{A^*}((x_1, \ldots, x_{j-1})R_q)$ with the image of x_j in R_q belonging to Q_q.

Here $Q \subseteq R$ is a prime in R containing q.

By Corollary O2, Q belongs to $\overline{A^*}((x_1, \ldots, x_{j-1})R)$.

Since $x_j \in Q$, this is a contradiction. The converse is similar.

Asymptotic sequences

Part (iii): By parts (i) and (ii), the given x_i form an asymptotic sequence if and only if their images in $\widehat{R}_z$ form an asymptotic sequence, for all minimal primes $z \subseteq \widehat{R}$.

Thus, we must prove that if R is a complete local domain, then $x_1, \ldots, x_r$ form an asymptotic sequence if and only if $\mathrm{height}(x_1, \ldots, x_r)R = r$, for all i.

For this, suppose $x_1, \ldots, x_r$ is an asymptotic sequence.

Since each x_i is chosen to avoid the primes in $\overline{A^*}((x_i, \ldots, x_{i-1})R)$, each x_i avoids the primes minimal over $(x_1, \ldots, x_{i-1})R$.

Therefore the ideals $(x_1, \ldots, x_i)R$ all have height i.

Conversely, suppose $x_1, \ldots, x_r$ generate an ideal having height r. The ideal generated by each $x_1, \ldots x_t$ has height t, for all $1 \leq t < r$ (since R is catenary).

If $x_1, \ldots, x_r$ do not form an asymptotic sequence, $x_j \in P$, for some $P \in \overline{A^*}((x_1, \ldots, x_{j-1})R)$, for some j.

By Proposition O3, $\mathrm{height}(P) \leq j - 1$.

Asymptotic sequences

On the other hand, $(x_1, \ldots, x_{j-1})R \subseteq P$, so $\mathrm{height}(P) \geq j-1$, and therefore $\mathrm{height}(P) = j-1$.

Since $x_j \in P$, this contradicts the assumption on the x_i.

Thus, $x_1, \ldots, x_r$ form an asymptotic sequence.

Part (iv): Follows immediately from part (iii).

Part (v): We use the obvious terminology:

We say that $x_1, \ldots, x_s$ form a maximal asymptotic sequence if they form an asymptotic sequence and there does not exist $y \in R$ such that $x_1, \ldots, x_s, y$ is an asymptotic sequence.

The second condition is equivalent to requiring $\mathfrak{m} \in \overline{A^*}((x_1, \ldots, x_s)R)$.

Asymptotic sequences

Set $\delta(R)$ to be the minimum of $\dim(\widehat{R}/z)$, taken over all minimal primes $z \subseteq \widehat{R}$.

By part (iii), the length of any asymptotic sequence is less than or equal to $\delta(R)$, including the length of a maximal asymptotic sequence.

Now suppose $x_1, \ldots, x_s$ is a maximal asymptotic sequence.

Then $\mathfrak{m} \in \overline{A^*}((x_1, \ldots, x_s)R)$. By parts (i) and (ii) above, there exists a minimal prime $z \subseteq \widehat{R}$ with $\mathfrak{m}\widehat{R}_z \in \overline{A^*}((x_1, \ldots, x_s)\widehat{R}_z)$.

By Proposition O3, $\mathrm{height}(\mathfrak{m}\widehat{R}_z) = \dim(R_z) \leq s$.

Thus, $\delta(R) \leq s$, which shows that all maximal asymptotic sequences in R have length $\delta(R)$. □

Quasi-unmixed local rings

We can now state and prove the characterization of quasi-unmixed local rings.

Theorem P3. Let $(R, \mathfrak{m})$ be a local ring. The following statements are equivalent.

(i) R is quasi-unmixed.

(ii) Every system of parameters forms an asymptotic sequence.

(iii) Some system of parameters forms an asymptotic sequence.

Proof. We let $\delta(R)$ have the same meaning as above. Set $d := \dim(R)$. If R is quasi-unmixed, then $\delta(R) = d$.

Let $x_1, \ldots, x_d$ be a sop and let I denote the ideal they generate. Then I is $\mathfrak{m}$-primary. It follows that the image of I in each $\widehat{R}_z$ is $\mathfrak{m}\widehat{R}_z$-primary for all minimal primes $z \subseteq \widehat{R}$.

Each $\widehat{R}_z$ has dimension d, therefore the images of $x_1, \ldots, x_d$ in each $\widehat{R}_z$ form a sop and thus generate an ideal of height d.

By Theorem H3, $x_1, \ldots, x_d$ is an asymptotic sequence. So, (i) implies (ii). Clearly (ii) implies (iii). If some sop forms an asymptotic sequence, this is clearly a maximal asymptotic sequence. The length of such equals $\delta(R)$ by Theorem H3. Thus $\delta(R) = \dim(R)$, and R is quasi-unmixed. $\square$

Quasi-unmixed local rings

As a corollary, we can prove one component of Ratliff's Theorem.

Corollary Q3. Let $(R, \mathfrak{m})$ be a local domain. If R satisfies the dimension formula, then R is quasi-unmixed.

Proof. By the previous theorem it suffices to show that R has a sop forming an asymptotic sequence. Suppose $x_1, \ldots, x_r$ is a maximal asymptotic sequence.

Then, $\mathfrak{m} \in \overline{A^*}(x_1, \ldots, x_r)R$.

On the other hand, by Proposition O3, $\mathrm{height}(\mathfrak{m}) \leq r$. Since $r \leq \mathrm{height}(\mathfrak{m})$, we must have $r = \mathrm{height}(\mathfrak{m}) = \dim(R)$. T his implies that $x_1, \ldots, x_r$ is a sop, and thus R is quasi-unmixed. □

We now want to work directly towards the other parts of Ratliff's theorem.

We start with two observations.

Observation 1. For Noetherian domains $A \subseteq B$ such that B is a finitely generated A-algebra, if A satisfies the dimension formula, then B satisfies the dimension formula.

To see this: Let C be a finitely generated B algebra. Then C is also a finitely generated A algebra. Let $Q \subseteq C$ be a prime ideal and set $P := Q \cap B$ and $P_0 := Q \cap A$. Then since A satisfies the dimension formula:

$$\text{height}(Q) + \text{tr.deg}_{k(P_0)} k(Q) = \text{height}(P_0) + \text{tr.deg}_A C,$$

and

$$\text{height}(P) + \text{tr.deg}_{k(P_0)} k(P) = \text{height}(P_0) + \text{tr.deg}_A B.$$

Solving each equation for $\text{height}(P_0$ and setting them equal to each other gives:

$$\text{height}(Q) + \text{tr.deg}_{k(P_0)} k(Q) - \text{tr.deg}_A C = \text{height}(P) + \text{tr.deg}_{k(P_0)} k(P) - \text{tr.deg}_A B.$$

Rewriting, we get:

$$\text{height}(Q) + \text{tr.deg}_{k(P_0)} k(Q) - \text{tr.deg}_{k(P_0)} k(P) = \text{height}(P) + \text{tr.deg}_A C - \text{tr.deg}_A B.$$

Additivity of transcendence degree gives:

$$\text{height}(Q) + \text{tr.deg}_{k(P)} k(Q) = \text{height}(P) + \text{tr.deg}_B C,$$

which is what we want.

Quasi-unmixed local rings

Observation 2. A Noetherian ring S is catenary if and only if for every pair of prime ideals $P \subseteq Q$, $\mathrm{height}(Q) = \mathrm{height}(P) + \mathrm{height}(Q/P)$.

To see this: suppose the height condition holds. Let $P \subseteq Q$ be prime ideals. To see that all saturated chains of primes between P and Q have the same length, we may mod out P and localize at Q.

Note that these operations preserve the height condition. Thus, we have to show that if the height condition holds, all maximal chains of primes in a local domain $(R, \mathfrak{m})$ have the same length,namely, $\dim(R)$.

Let $(0) \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_s = \mathfrak{m}$ be a maximal chain of length s. Clearly $\mathrm{height}(Q_1) = 1$.

By the height condition
$\mathrm{height}(Q_2) = \mathrm{height}(Q_2/Q_1) + \mathrm{height}(Q_1) = 1 + 1 = 2$, since, by assumption, there are no primes between Q_1 and Q_2.

Continuing in this fashion, we see $\mathrm{height}(Q_i) = i$, for all i.

Thus, $s = \mathrm{height}(Q_s) = \mathrm{height}(\mathfrak{m}) = \dim(R)$, which is what we want.

The converse is clear.

We can now state and prove a second implication in Ratliff's Theorem.

Proposition R3. Let R be a universally catenary Noetherian domain. Then R satisfies the dimension formula.

Proof. We just have to prove: If T is a Noetherian domain, and $T = R[x]$, for some $x \in T$, then the dimension formula holds between R and T. If x is algebraically independent over R, then we have verified the dimension formula in this case in Remark (iii) following the definition of the dimension formula.

Suppose x is algebraic over R. Let A denote the polynomial ring in one variable over R set K to be the kernel of the natural homomorphism from A to T.

Since $\mathrm{tr.deg}_R T = 0$, we must show

$$\mathrm{height}(Q) + \mathrm{tr.deg}_{k(P)} k(Q) = \mathrm{height}(P).$$

Let Q_0 denote the preimage of Q in A, so that $Q = Q_0/K$.

Since A is catenary,

$$\mathrm{height}(Q_0) = \mathrm{height}(Q_0/K) + \mathrm{height}(K) = \mathrm{height}(Q) + 1. \quad (*)$$

Quasi-unmixed local rings

Since the dimension formula holds between A and R we have

$$\mathrm{height}(Q_0) + \mathrm{tr.deg}_{k(P)} k(Q_0) = \mathrm{height}(P) + \mathrm{tr.deg}_R A = \mathrm{height}(P) + 1.$$

Using (*) in this last equation we have

$$\mathrm{height}(Q) + 1 + \mathrm{tr.deg}_{k(P)} k(Q_0) = \mathrm{height}(P) + 1. \quad (**)$$

But $A/Q_0 = T/Q$, so $\mathrm{tr.deg}_{k(P)} k(Q_0) = \mathrm{tr.deg}_{k(P)} k(Q)$.

Substituting this into (**) and cancelling 1 yields

$$\mathrm{height}(Q) + \mathrm{tr.deg}_{k(P)} k(Q) = \mathrm{height}(P),$$

which is what we want. □

Cohen-Macaulay implies catenary

Here is a result of independent interest that plays a key role in our analysis.

Proposition S3. Let S be a a Cohen-Macaulay ring. Then S is catenary.

Proof. We just have to check the height condition in the observation above. Let $P \subseteq Q$ be primes. We may assume that S is local at Q.

Suppose P has height h and set $d =: \dim(S)$. Take $\underline{x} = x_1, \ldots, x_h$ a maximal regular sequence from P.
Then

$$\dim(S) - \mathrm{height}(P) = d - h = \mathrm{depth}(S/(\underline{x})) \leq \dim(R/P),$$

the latter inequality holds since P is an associated prime of $S/(\underline{x})$.

On the other hand,

$$\dim(R/P) + \mathrm{height}(P) \leq \dim(S)$$

always holds in a local ring,and thus,

$\dim(S) = \mathrm{height}(P) + \dim(S/P)$, which is what we want. □

Quasi-unmixed local rings

Proposition T3. Let $(R, \mathfrak{m})$ be a complete local domain. Then R is universally catenary and satisfies the dimension formula.

Proof. We use the fact that a homomorphic image of a catenary ring is catenary. To see that R is universally catenary, it suffices to show that a polynomial ring in finitely many variables over R is catenary.

By Cohen's Structure Theorem, R is the homomorphic image of a regular local ring S. Hence any polynomial ring B over R is a homomorphic image of a polynomial ring A over S.

Since S is Cohen-Macaulay, A is Cohen-Macaulay, and therefore catenary.

Thus, B is catenary, which shows R is universally catenary.

The second statement is now immediate from Proposition R3, $\square$

Remark. Since the catenary property does not require the ring in question to be an integral domain, the proof above shows that a complete local ring is catenary.